Monotone Class Theorem & Product Measures

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The monotone class theorem is often presented rather confusingly. I write this document in an attempt to clarify the concept (mostly to myself, but to anyone else who wishes to read this too).

I. DEFINITIONS

Definition 1. Here we denote the *power set* of X by $\mathcal{P}(X)$, the set of all subsets of X.

Definition 2. An algebra of sets $\mathcal{D} \subseteq \mathcal{P}(X)$ is a collection of subsets of X with the following properties:

1. $A \in \mathcal{D} \iff A^c \in \mathcal{D}$

2. $\varnothing$ (or X) $\in \mathcal{D}$

3. Closure under finite unions.

Definition 3. A monotone class is a family of sets $\mathcal{M} \subseteq \mathcal{P}(X)$, so that for all monotone sequences of sets $A_1 \subseteq A_2 \ldots, A_i \in \mathcal{M}$, we have $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$, as well as if $B_1 \supseteq B_2 \ldots$, then $\bigcap_{i=1}^{\infty} B_i \in \mathcal{M}$.

Definition 4. A σ-algebra $\mathcal{A}$ is a family of subsets $M \subseteq \mathcal{P}(X)$ with the following properties

1. $X, \varnothing \in \mathcal{A}$

2. $A \in \mathcal{A} \iff A^c \in \mathcal{A}$

3. Countable unions of elements in $\mathcal{A}$ are also elements of $\mathcal{A}$.

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II. ADDITIONAL PROPERTIES

We state the following well known theorem without proof (in any case, it is not difficult to prove).

Theorem 5. *The final condition, together with the second condition, is equivalent to countable intersections of elements in $\mathcal{A}$ being also in $\mathcal{A}$.*

Corollary 6. *It is now obvious that all σ-algebras are also monotone classes.*

Theorem 7. *Algebras are closed under set differences.*

Proof. Let $\mathcal{D} \subseteq \mathcal{P}(X)$ be an algebra of sets, and let A, B be sets in $\mathcal{D}$. It follows that

$$A \backslash B = A \cap B^c.$$

which is in $\mathcal{D}$. □

III. THE THEOREM

The goal of this will be to prove the *Monotone Class Theorem*:

Theorem 8. *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ be an algebra of subsets of X. Then the smallest monotone class containing $\mathcal{D}$ ist also the sigma-algebra generated by $\mathcal{D}$, or the smallest σ-algebra containing $\mathcal{D}$.*

Remark 9. *What is exactly the problem? The problem is that not all monotone classes are σ-algebras. Consider, for example, the following monotone class on $X = \{1, 2, 3\}$:*

$$\mathcal{M} = \{\varnothing, \{1\}, \{2\}, \{3\}, X\}.$$

Clearly, it is trivially a monotone class, but not a sigma algebra.

Here, the main problem is that finite unions are not elements of the monotone class. We will see eventually that this is the only *problem. Near the end, we will consider countable unions $\cup_{i=1}^{\infty} A_i$ by setting $B_i = \cup_{j=1}^{i} A_i$, which is a monotone sequence and using the monotone union property.*

We now embark on the

Proof. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. We define $\mathcal{A}$ to be the smallest σ-algebra containing $\mathcal{C}$ and $\mathcal{M}$ to be the smallest monotone class containing $\mathcal{C}$.

Since all σ-algebras are monotone classes, then $\mathcal{M} \subseteq \mathcal{A}$. It remains to prove the opposite inclusion.

Let $P \subseteq X$ be a subset of X. We define a set $\Omega(P)$ to be the class of sets $Q \subseteq X$ such that

$$\Omega(P) = \{Q | Q \subseteq X, P - Q \in \mathcal{M}, Q - P \in \mathcal{M}, P \cup Q \in \mathcal{M}\}.$$

Clearly, since $\mathcal{M}$ is a monotone class, $\Omega(P)$ is also a monotone class for all $P \subseteq X$. The symmetry also shows that

$$Q \in \Omega(P) \iff P \in \Omega(Q).$$

Now we fix $P \in \mathcal{D}$. Recall that $\mathcal{D} \subseteq \mathcal{M}$. Thus, by the closure of the algebra $\mathcal{D}$ under set differences, we have $\mathcal{D} \subseteq \Omega(P)$ for all $P \in \mathcal{D}$. Since $\Omega(P)$ is a monotone class, it follows that

$$\mathcal{M} \subseteq \Omega(P) \; \forall P \in \mathcal{D}.$$

Thus all elements in $\mathcal{M}$ also have the defining properties of $\Omega(P)$. In particular, we conclude that $\mathcal{M}$ is closed under finite differences and unions. Since $\mathcal{D}$ is an algebra, we have $X \in \mathcal{M}$. The closure under complement property follows then from closure under differences.

The final step is as promised: We consider a countable union $\bigcup_{i=1}^{\infty} A_i, A_i \in \mathcal{M}$. We set $B_i = \cup_{j=1}^{i} A_i$. Every set B_i is in $\mathcal{M}$ because of the closure of finite unions, and each B_i is included in the next B_{i+1}. Applying the monotone property, we see that $\mathcal{M}$ is closed under countable unions, or $\mathcal{M}$ is a sigma algebra. However, since $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{D}$, we have $\mathcal{A} \subseteq \mathcal{M}$. The equality follows. □

IV. APPLICATIONS

Definition 10. We consider two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$. We wish to construct a measure and σ-algebra on the product $X \times Y$. We define a *measurable rectangle* to be a set of the form $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The set of *elementary sets* $\mathcal{E}$ is the set of all finite unions of disjoint measurable rectangles.

Theorem 11. *$\mathcal{E}$ is an algebra*

Proof. Closure under intersections and finite differences follows from

$$A_1 \times B_1 \cap A_2 \times B_2 = (A_1 \cap A_2) \times (B_1 \cap B_2)$$
$$A_1 \times B_1 \backslash A_2 \times B_2 = [(A_1 \backslash A_2) \times B_1] \cap [(A_1 \cap A_2) \times (B_1 \backslash B_2)]$$

Closure under finite unions follow from

$$P \cup Q = (P - Q) \cup Q.$$

□